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MATHEMATICAL STUDY OF THE STABILITY OF FIXED POINTS OF SYSTEMS OF DIFFERENTIAL EQUATIONS DESCRIBING BIOCHEMICAL PROCESSES RATES

Mathematical study of the stability of fixed points of systems of differential equations describing biochemical processes rates is performed in the article. A system of differential equations for deviations is constructed, which describes the behavior of the system near the fixed point. An analysis of the general solution of the system of differential equations describing biochemical processes rates is made. The behavior of the systems near fixed points is investigated by the method of small perturbations. The conditions of existence of the limit cycle of the system of differential equations are investigated. For the cases when the system of differential equations cannot have an analytical solution, integrated curves are constructed by qualitative research. Phase trajectories of the system of differential equations describing biochemical processes rates are constructed. The definition of the nature of the stability of fixed points is considered and investigated.

Keywords: *concentration of substance, system of differential equations, limit cycle.*

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МАТЕМАТИЧНЕ ДОСЛІДЖЕННЯ СТІЙКОСТІ ОСОБЛИВИХ ТОЧОК СИСТЕМ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ, ЯКІ ОПИСУЮТЬ ШВИДКОСТІ БІОХІМІЧНИХ ПРОЦЕСІВ

У статті виконано математичне дослідження стійкості особливих точок систем диференціальних рівнянь, які описують швидкості біохімічних процесів. Розглянуто і досліджено визначення характеру стійкості особливих точок.

Ключові слова: *концентрація речовини, система диференціальних рівнянь, граничний цикл.*

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МАТЕМАТИЧЕСКОЕ ИССЛЕДОВАНИЕ УСТОЙЧИВОСТИ ОСОБЫХ ТОЧЕК СИСТЕМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ, ОПИСЫВАЮЩИХ СКОРОСТИ БИОХИМИЧЕСКИХ ПРОЦЕССОВ

В статье выполнено математическое исследование устойчивости особых точек систем дифференциальных уравнений, описывающих скорости биохимических процессов. Рассмотрено и исследовано определение характера устойчивости особых точек.

Ключевые слова: *концентрация вещества, система дифференциальных уравнений, предельный цикл.*

Problem formulation. Biochemical processes rates in their mathematical modeling are described by the system of differential equations, which can be written in the form

$$\frac{dc_i(t)}{dt} = f_i(c_1, c_2, \ldots, c_N), \quad i = \overline{1, N}, \tag{1}$$

where c_i – concentrations of substances in biochemical reactions; $c_i \geq 0$.

The necessity and methods of reducing the number of equations to two or three differential equations in the system (1) are investigated in the article [1].

For a mathematical study of the stability of fixed points, we consider a system of two differential equations [2]:

$$\begin{cases} \dfrac{dc_1(t)}{dt} = f_1(c_1, c_2), \\ \dfrac{dc_2(t)}{dt} = f_2(c_1, c_2). \end{cases} \tag{2}$$

Analysis of the latest research and publications. We consider how in the general case the nature of the stability of a fixed point is determined.

In a mechanical system, it is necessary to make a slight push or shift from the equilibrium position and see if the system returns to this equilibrium position. The same should be done in mathematical modeling of dynamic systems [3]-[6].

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Presentation of the main material. Let $\overline{c}_1, \overline{c}_2$ be the coordinates of a fixed point (equilibrium position) of the system of differential equations (2). Let us define a small deviation $\Delta c_1 \ll \overline{c}_1$, $\Delta c_2 \ll \overline{c}_2$ from the equilibrium position and substitute it into system of differential equations (2) $c_1 = \overline{c}_1 + \Delta c_1$, $c_2 = \overline{c}_2 + \Delta c_2$. Then we decompose the right-hand sides of the differential equations of the system (2) into Taylor series in the neighborhood of the point $(\overline{c}_1;\overline{c}_2)$ (i.e. in the neighborhood of the equilibrium position), limited (taking into account the smallness $\Delta c_1, \Delta c_2$) to the first terms of the series (terms of the first degree):

$$f_1(c_1,c_2) = f_1(\overline{c}_1+\Delta c_1, \overline{c}_2+\Delta c_2) = f_1(\overline{c}_1,\overline{c}_2) + \Delta c_1 \left.\frac{\partial f_1(c_1,c_2)}{\partial c_1}\right|_{(\overline{c}_1;\overline{c}_2)} + \Delta c_2 \left.\frac{\partial f_1(c_1,c_2)}{\partial c_2}\right|_{(\overline{c}_1;\overline{c}_2)} + \cdots,$$

$$f_2(c_1,c_2) = f_2(\overline{c}_1+\Delta c_1, \overline{c}_2+\Delta c_2) = f_2(\overline{c}_1,\overline{c}_2) + \Delta c_1 \left.\frac{\partial f_2(c_1,c_2)}{\partial c_1}\right|_{(\overline{c}_1;\overline{c}_2)} + \Delta c_2 \left.\frac{\partial f_2(c_1,c_2)}{\partial c_2}\right|_{(\overline{c}_1;\overline{c}_2)} + \cdots.$$

Thus, in order to study the equilibrium position (the fixed point $(\overline{c}_1;\overline{c}_2)$) of the non-linear system of differential equations (2), we, at the point $(\overline{c}_1;\overline{c}_2)$, located on the phase plane $O'c_1c_2$, placed the origin *O* of the phase plane $O\Delta c_1\Delta c_2$ (Fig. 1) and decomposed the functions $f_1(c_1,c_2)$ and $f_2(c_1,c_2)$ into Taylor series in the neighborhood of the point $(\overline{c}_1;\overline{c}_2)$.

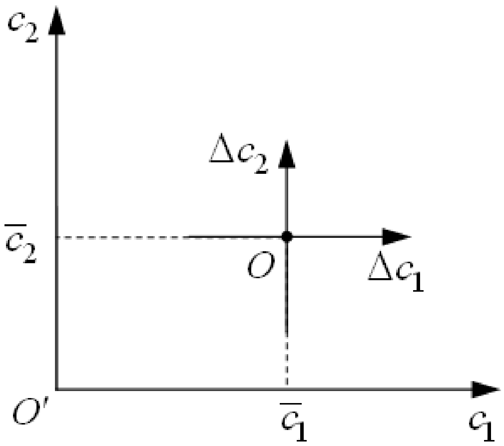

Fig. 1. **The fixed point** $(\overline{c}_1;\overline{c}_2)$

Since the derivatives of constants $\frac{d\overline{c}_1}{dt} = 0$, $\frac{d\overline{c}_2}{dt} = 0$ and $\Delta c_1, \Delta c_2$ are variables, i.e. $\Delta c_1 = \Delta c_1(t), \Delta c_2 = \Delta c_2(t)$, then left-hand sides of the differential equations of the system (2) take on the form:

$$\frac{d\left(c_1(t)\right)}{dt} = \frac{d\left(\overline{c}_1 + \Delta c_1(t)\right)}{dt} = \frac{d\left(\Delta c_1(t)\right)}{dt},$$
$$\frac{d\left(c_2(t)\right)}{dt} = \frac{d\left(\overline{c}_2 + \Delta c_2(t)\right)}{dt} = \frac{d\left(\Delta c_2(t)\right)}{dt}.$$

Since at the fixed (stationary) point $(\overline{c}_1;\overline{c}_2)$ $\frac{d\overline{c}_1}{dt} = 0$ and $\frac{d\overline{c}_2}{dt} = 0$, then from the system (2) we obtain

$$f_1(\bar{c}_1, \bar{c}_2) = 0, \quad f_2(\bar{c}_1, \bar{c}_2) = 0.$$

Then, up to the terms of the second order of smallness, after substitution into the system of differential equations (2), we obtain the system of differential equations

$$\begin{cases} \dfrac{d(\Delta c_1)}{dt} = \Delta c_1 \left.\dfrac{\partial f_1(c_1, c_2)}{\partial c_1}\right|_{(\bar{c}_1;\bar{c}_2)} + \Delta c_2 \left.\dfrac{\partial f_1(c_1, c_2)}{\partial c_2}\right|_{(\bar{c}_1;\bar{c}_2)}, \\ \dfrac{d(\Delta c_2)}{dt} = \Delta c_1 \left.\dfrac{\partial f_2(c_1, c_2)}{\partial c_1}\right|_{(\bar{c}_1;\bar{c}_2)} + \Delta c_2 \left.\dfrac{\partial f_2(c_1, c_2)}{\partial c_2}\right|_{(\bar{c}_1;\bar{c}_2)}. \end{cases} \quad (3)$$

Denoting

$$\left.\frac{\partial f_1(c_1, c_2)}{\partial c_1}\right|_{(\bar{c}_1;\bar{c}_2)} = \alpha_1, \quad \left.\frac{\partial f_1(c_1, c_2)}{\partial c_2}\right|_{(\bar{c}_1;\bar{c}_2)} = \alpha_2, \quad \left.\frac{\partial f_2(c_1, c_2)}{\partial c_1}\right|_{(\bar{c}_1;\bar{c}_2)} = \beta_1, \quad \left.\frac{\partial f_2(c_1, c_2)}{\partial c_2}\right|_{(\bar{c}_1;\bar{c}_2)} = \beta_2,$$

in the system of differential equations (3), we write the system (2) in the form

$$\begin{cases} \dfrac{dc_1}{dt} = \dfrac{d(\Delta c_1)}{dt} = \alpha_1 \Delta c_1 + \alpha_2 \Delta c_2, \\ \dfrac{dc_2}{dt} = \dfrac{d(\Delta c_2)}{dt} = \beta_1 \Delta c_1 + \beta_2 \Delta c_2 \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{d(\Delta c_1)}{dt} = \alpha_1 \Delta c_1 + \alpha_2 \Delta c_2, \\ \dfrac{d(\Delta c_2)}{dt} = \beta_1 \Delta c_1 + \beta_2 \Delta c_2. \end{cases} \quad (4)$$

This system of differential equations for deviations (perturbations) $\Delta c_1(t)$, $\Delta c_2(t)$ describes the behavior of the system near the fixed point.

We find the general solution of the system of differential equations (4) in the form

$$\Delta c_1 = Ae^{kt}, \; \Delta c_2 = Be^{kt}. \quad (5)$$

Substituting (5) into (4), we obtain

$$\begin{cases} Ake^{kt} = \alpha_1 Ae^{kt} + \alpha_2 Be^{kt}, \\ Bke^{kt} = \beta_1 Ae^{kt} + \beta_2 Be^{kt} \end{cases}$$

or reducing each equation of the system by the factor e^{kt}, we obtain the system of algebraic equations

$$\begin{cases} Ak = \alpha_1 A + \alpha_2 B, \\ Bk = \beta_1 A + \beta_2 B. \end{cases}$$

From the second equation of this system of algebraic equations we find $A = B\dfrac{k-\beta_2}{\beta_1}$ and substitute it into the first equation, and reduce the obtained equation by the factor *B*, taking into account that the amplitude $B \neq 0$, we obtain the characteristic equation

$$(k-\alpha_1)(k-\beta_2) = \alpha_2\beta_1 \;\text{ or }\; k^2 - (\alpha_1+\beta_2)k + \alpha_1\beta_2 - \alpha_2\beta_1 = 0, \quad (6)$$

whence

$$k_{1,2} = \frac{\alpha_1 + \beta_2 \pm \sqrt{(\alpha_1 + \beta_2)^2 - 4(\alpha_1\beta_2 - \alpha_2\beta_1)}}{2} =$$
$$= \frac{\alpha_1 + \beta_2 \pm \sqrt{(\alpha_1 - \beta_2)^2 + 4\alpha_2\beta_1}}{2} = \frac{\alpha_1 + \beta_2 \pm \sqrt{D}}{2}. \tag{7}$$

Then the general solution of the system of differential equations (4) can be written in the form

$$\Delta c_1 = A_{\mathrm{I}} e^{k_1 t} + A_{\mathrm{II}} e^{k_2 t}, \quad \Delta c_2 = B_{\mathrm{I}} e^{k_1 t} + B_{\mathrm{II}} e^{k_2 t}, \tag{8}$$

where the amplitudes $A_{\mathrm{I}}, A_{\mathrm{II}}, B_{\mathrm{I}}, B_{\mathrm{II}}$ depend on the initial data.

The values k_1 and k_2 determine the nature of motion near the fixed point.

Consider combinations of values k_1 and k_2.

1) Discriminant of the characteristic equation (6)

$$D = (\alpha_1 - \beta_2)^2 + 4\alpha_2\beta_1 \geq 0.$$

Then the two roots are valid. In this case, there may be three cases:

1a. The roots k_1 and k_2 are negative. Then the solution (8) will be in the form of decreasing exponents over time, that is the fixed point is stable. All phase trajectories tend to the fixed point over time.

1b. The roots k_1 and k_2 are positive. Then the fixed point is unstable. The phase point from arbitrary initial data moves away from the fixed point.

1c. The roots k_1 and k_2 have different signs, for example, $k_1 > 0$ and $k_2 < 0$. Then the fixed point is unstable, since the term with a positive exponent will always prevail over time. However, in some cases, when the initial data are such that $A_{\mathrm{I}} = B_{\mathrm{I}} = 0$, then Δc_1 and Δc_2 will decrease over time, since from (8) we obtain

$$\Delta c_1 = A_{\mathrm{II}} e^{k_2 t}, \quad \Delta c_2 = B_{\mathrm{II}} e^{k_2 t}.$$

Then

$$\frac{\Delta c_2}{\Delta c_1} = \frac{B_{\mathrm{II}}}{A_{\mathrm{II}}} \text{ or } \Delta c_2 = \frac{B_{\mathrm{II}}}{A_{\mathrm{II}}} \Delta c_1. \tag{9}$$

The equation (9) is the equation if the straight line on the phase plane: along this straight line the phase point goes to the fixed point *O* (Fig. 2).

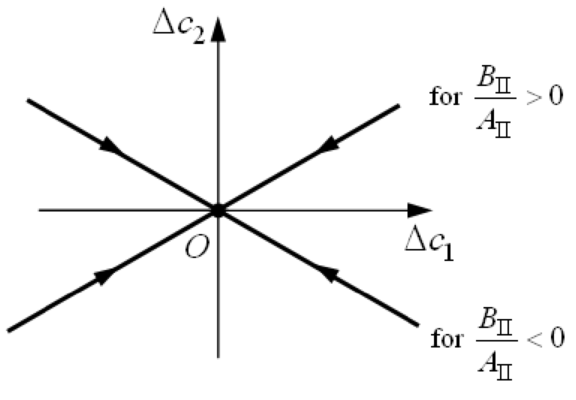

Fig. 2. **The fixed point *O* for** $k_1 > 0$, $k_2 < 0$, $A_{\mathrm{I}} = B_{\mathrm{I}} = 0$

2) The discriminant of the characteristic equation (6) has the form

$$D = (\alpha_1 - \beta_2)^2 + 4\alpha_2\beta_1 < 0.$$

Then from the system of differential equations (4) we obtain a differential equation of the second order.

For this equation to be obtained, we differentiate one of two, for example, the first differential equation of the system (4).

We substitute the right-hand side of the second differential equation of the system (4) for $\frac{d(\Delta c_2)}{dt}$ into the differentiated first differential equation and, finding Δc_2 form the first differential equation of the system (4), we substitute Δc_2 into the differentiated equation. Thus, we obtain the differential equation of the second order:

$$\frac{d^2(\Delta c_1)}{dt^2} + 2\gamma\frac{d(\Delta c_1)}{dt} + \omega_0^2\Delta c_1 = 0, \qquad (10)$$

where constant values γ and ω_0 are expressed in terms of the coefficients α, β :

$$2\gamma = -(\alpha_1 + \beta_2), \quad \omega_0^2 = \alpha_1\beta_2 - \alpha_2\beta_1.$$

Then we write the characteristic equation (6) in the form

$$k^2 + 2\gamma k + \omega_0^2 = 0,$$

whence

$$k_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

Consider the case with the negative discriminant of this characteristic equation, i.e. $\omega_0^2 > \gamma^2$. In this case, we obtain the complex-conjugate roots of this equation

$$k_{1,2} = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = -\gamma \pm i\omega,$$

where $\omega^2 = \omega_0^2 - \gamma^2$.

Then we write the general solution of the differential equation (10) in the form

$$\Delta c_1(t) = e^{-\gamma t}(A_1 \cos\omega t + A_2 \sin\omega t). \qquad (11)$$

Note that equation (10) is a linear differential equation of motion (or the equation of a damped harmonic oscillator). Its solution is characterized by trigonometric (or harmonic) oscillations, damping for $\gamma > 0$.

From the first differential equation of the system (4) we find Δc_2 :

$$\Delta c_2 = \frac{\frac{d(\Delta c_1)}{dt} - \alpha_1\Delta c_1}{\alpha_2}. \qquad (12)$$

Differentiating the solution (11), we obtain

$$\frac{d(\Delta c_1)}{dt} = e^{-\gamma t}\left((\omega A_2 - \gamma A_1)\cos\omega t - (\omega A_1 + \gamma A_2)\sin\omega t\right). \qquad (13)$$

Substituting (13) and (11) into (12), we obtain

$$\Delta c_2 = e^{-\gamma t}\left(\frac{\omega A_2 - (\gamma + \alpha_1)A_1}{\alpha_2}\cos\omega t - \frac{\omega A_1 + (\gamma + \alpha_1)A_2}{\alpha_2}\sin\omega t\right)$$

or

$$\Delta c_2(t) = e^{-\gamma t}(B_1 \cos \omega t + B_2 \sin \omega t), \tag{14}$$

where

$$B_1 = \frac{\omega A_2 - (\gamma + \alpha_1) A_1}{\alpha_2}, \quad B_2 = -\frac{\omega A_1 + (\gamma + \alpha_1) A_2}{\alpha_2}.$$

Thus, the general solution of the system of differential equations (4) consists of general solutions (11) and (14) and has the form

$$\begin{aligned} \Delta c_1(t) &= e^{-\gamma t}(A_1 \cos \omega t + A_2 \sin \omega t), \\ \Delta c_2(t) &= e^{-\gamma t}(B_1 \cos \omega t + B_2 \sin \omega t), \end{aligned} \tag{15}$$

where ω circular frequency $\Delta c_1(t)$ and $\Delta c_2(t)$.

The graph of the function (11) is shown in Fig. 3 for $\gamma > 0$.

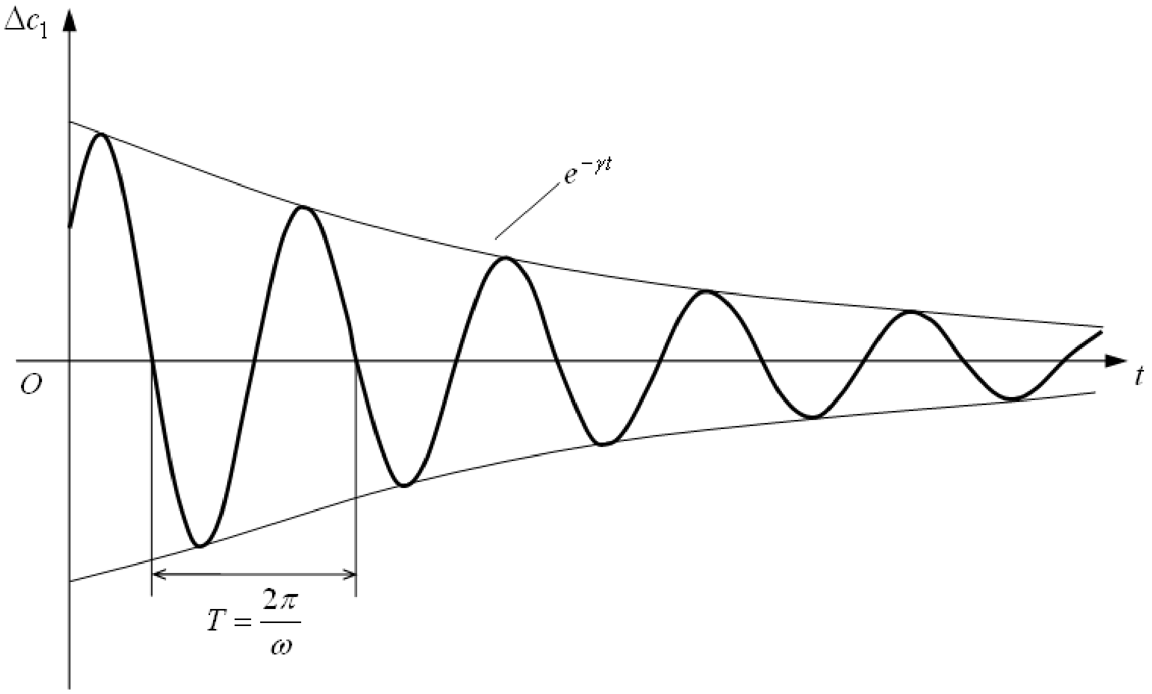

Fig. 3. **The graph of the function (11) for $\gamma > 0$**

The graph of the function (14) has a similar form.

From the general solution (15) we see that the nature of the stability of the fixed point depends only on the value and sign of γ. Consider the following cases:

2a. $\gamma = 0$. Then $\Delta c_1(t)$, $\Delta c_2(t)$ are harmonic functions of time, i.e. when defining the initial deviations $\Delta c_{1(0)}$, $\Delta c_{2(0)}$ there are undamped oscillations with frequency ω in the system. On the phase plane $O\Delta c_1 \Delta c_2$, different graphs (closed elliptical curves that are nested) will correspond to different initial data. In this case, the fixed point (the origin *O*) is a center, and the phase portrait consists of a continuum of concentric elliptical closed curves. Repeated (periodic) motion occurs in the system.

We write the solution (15) for $\gamma = 0$:

$$\Delta c_1(t) = A_1 \cos \omega t + A_2 \sin \omega t, \tag{16}$$

$$\Delta c_2(t) = B_1 \cos \omega t + B_2 \sin \omega t. \tag{17}$$

Then (16) and (17) can be written in the form

$$\Delta c_1(t) = A_1 \cos \omega t + A_2 \sin \omega t = A\cos(\omega t + \varphi_1), \tag{18}$$

$$\Delta c_2(t) = B_1 \cos \omega t + B_2 \sin \omega t = B\sin(\omega t + \varphi_2), \tag{19}$$

where

$$\varphi_1 = \operatorname{arctg}\left(-\frac{A_2}{A_1}\right), \quad A = \frac{A_1}{\cos \varphi_1}; \tag{20}$$

$$\varphi_2 = \operatorname{arctg}\left(\frac{B_1}{B_2}\right), \quad B = \frac{B_1}{\sin \varphi_2}. \tag{21}$$

Let us show how to obtain the equality (18). From the expressions (20) we have

$$\operatorname{tg}\varphi_1 = -\frac{A_2}{A_1} \quad \text{or} \quad \frac{\sin \varphi_1}{\cos \varphi_1} = -\frac{A_2}{A_1}, \quad A_1 = A\cos \varphi_1.$$

Then $\frac{\sin \varphi_1}{\cos \varphi_1} = -\frac{A_2}{A\cos \varphi_1}$, whence $A_2 = -A\sin \varphi_1$. Substituting A_1 and A_2 into (18), we obtain

$$\Delta c_1(t) = A\left(\cos \varphi_1 \cos \omega t - \sin \varphi_1 \sin \omega t\right) = A\cos(\omega t + \varphi_1).$$

Similarly, we show how to obtain the equality (19). From the expressions (21) we have

$$\operatorname{tg}\varphi_2 = \frac{B_1}{B_2} \quad \text{or} \quad \frac{\sin \varphi_2}{\cos \varphi_2} = \frac{B_1}{B_2}, \quad B_1 = B\sin \varphi_2.$$

Then $\frac{\sin \varphi_2}{\cos \varphi_2} = \frac{B\sin \varphi_2}{B_2}$, whence $B_2 = B\cos \varphi_2$. Substituting B_1 and B_2 into (19), we obtain

$$\Delta c_2(t) = B\left(\sin \varphi_2 \cos \omega t + \cos \varphi_2 \sin \omega t\right) = B\sin(\omega t + \varphi_2).$$

Consider the case when $\varphi_1 = \varphi_2 = \varphi.$ Then (18) and (19) can be written in the form

$$\Delta c_1(t) = A\cos(\omega t + \varphi), \tag{22}$$

$$\Delta c_2(t) = B\sin(\omega t + \varphi). \tag{23}$$

In this case, the phase trajectories are a family of concentric ellipses with the center at the fixed point *O* (the phase portrait of the system has the form of the continuum of concentric ellipses – Fig. 4):

$$\frac{\left(\Delta c_1(t)\right)^2}{A^2} + \frac{\left(\Delta c_2(t)\right)^2}{B^2} = 1.$$

There is periodic motion in the system. Each point (except for the point *O*) is passed in time $T = \frac{2\pi}{\omega}$ again. The coordinates are periodic on *t* with the period *T* and are determined by the formulas (22) and (23). The center (the fixed point *O*) is a stable equilibrium position of the system (2).

Note that, for example, when $A > 0, B > 0, \varphi = 0$ with increasing t from $t = 0$ to $t = \frac{\frac{\pi}{2}}{\omega}$ $\Delta c_1(t) = A\cos(\omega t + \varphi)$ decreases, and $\Delta c_2(t) = B\sin(\omega t + \varphi)$ increases. This makes it possible to determine the direction of all trajectories (see quadrant *I* of Fig. 4).

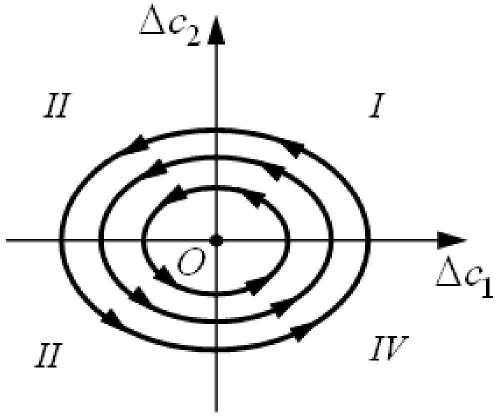

Fig. 4. **The phase portrait of the system for** $\varphi_1 = \varphi_2 = \varphi$.

2b. $\gamma < 0$. Then the presence of the factor $e^{-\gamma t}$ in the solution (15), which $e^{-\gamma t} \to \infty$ as $t \to \infty$, leads to the family of spirals that unwind from around the point *O*, not intersecting, on the phase plane $O\Delta c_1\Delta c_2$. In this case, the fixed point (the point *O*) is an unstable focus, and the phase portrait consists of repelling spirals.

2c. $\gamma > 0$. Then the presence of the factor $e^{-\gamma t}$ in the solution (15), which $e^{-\gamma t} \to 0$ as $t \to \infty$, leads to the representation of deviations $\Delta c_1(t) \to 0,\ \Delta c_2(t) \to 0$ in the form of damped oscillations. That is, the system performs damping, free (no external influence) oscillations. On the phase plane $O\Delta c_1\Delta c_2$, the graphs are the family of twisting spirals tending to point *O*. In this case, the fixed point *O* is a stable focus, and the phase portrait consists of attracting spirals.

The circular frequency ω of rotation of a point in a spiral is inversely proportional to the period of one rotation of a point in a spiral: $\omega = \frac{2\pi}{T}$, and the value γ shows how quickly the spiral unwinds or twists.

Thus, from points 2b, 2c, taking into account point 2a, it follows that the transition γ through zero for the solution (15) causes bifurcation of the entire phase portrait, i.e. its qualitative change.

Therefore, due to the method of small perturbations it is possible to define the nature of the stability of fixed points, i.e. to investigate the behavior of the system near fixed points.

It is also very important to know how systems behave far away from fixed points.

If time $t \to \infty$, then in the case of unstable nodes or foci, the phase point over time *t* moves away from a fixed point far enough, where it is no longer possible to use linearized systems of equations, which we obtain assuming small deviations Δc_1 and Δc_2. Therefore, the study of the behavior of the system far from the fixed point is carried out by geometric construction of integral curves.

In practice, no real value, including the concentration of a chemical substance, can grow indefinitely. At some point in the system itself there will be conditions that limit the growth of these values. However, systems of differential equations of type (1) can have stable fixed points at infinity. This happens when some important limitation is not taken into account.

Note that trajectories can proceed from an unstable fixed point when the initial conditions correspond to the equilibrium position (the point *O*)

Obviously, the trajectories that emerge from an unstable fixed point must go somewhere:

a) There is a stable position of equilibrium, to which all trajectories tend, near the unstable one;

b) There is no stable point nearby, but the trajectories do not go to infinity.

In this case, there is at least one closed phase curve, to which the phase trajectories should go in the limit. This curve is the limit cycle.

A limit cycle is the only closed orbit that is in a ring neighborhood (in a three-dimensional one, in a tubular neighborhood), that is, if there is a ring neighborhood that does not contain other closed trajectories except for the limit cycle. Thus, the limit cycle is isolated from all other closed trajectories. The sufficient condition for the existence of the limit cycle of the system (2) is the principle of the ring: if a ring

$$R_1^2 \le (c_1 - \bar{c}_1)^2 + (c_2 - \bar{c}_2)^2 \le R_2^2,$$

can be found on the phase plane $O'c_1c_2$ such that all system trajectories that begin at the boundary of this ring enter inside the ring or all of them exit it simultaneously, then there is a limit cycle inside the ring.

If the trajectories are wound on the limit cycle on two sides as $t \to \infty$, then the limit cycle is stable (attractive) or attractor.

If the trajectories are spirals that move away from the limit cycle on two sides as $t \to \infty$, then the limit cycle is unstable (repulsive) or repellent.

If the trajectories on one side are wound on the limit cycle as $t \to \infty$ and move away from it on the other side as $t \to \infty$, then the limit cycle is semi-stable.

Thus, the phase portrait, which, for example, contains a stable limit cycle is characterized by the presence of an annular neighborhood such that all trajectories that cross the boundary of this neighborhood, go to the limit cycle as $t \to \infty$.

The system may not have limit cycles.

Limit cycles do not always look like a circle.

Limit cycles can often be detected by passing to polar coordinates.

There may be more than one limit cycle. It depends on the complexity of the nonlinear functions $f_1(c_1, c_2)$, $f_2(c_1, c_2)$.

Since the limit trajectory is closed, it must be a periodic motion. We can consider three-dimensional space $O\Delta c_1 \Delta c_2 t$, in which the projection of motion on the phase plane $O\Delta c_1 \Delta c_2$ is a phase trajectory, and projections on the planes $O\Delta c_1 t$ and $O\Delta c_2 t$ are the sweeps of the process in time.

Then the limit with a phase trajectory winding on it is constructed. The motion of the phase point in the cycle corresponds to the oscillations $\Delta c_1(t)$ and $\Delta c_2(t)$ with constant amplitudes, which are set approximately in the second period after the start of motion.

The stability of the amplitude of oscillations ensures the stability of the limit cycle, which is the self-oscillating mode, i.e. oscillations occur without periodic external influences and can be maintained for as long as possible, but with an energy source such as sunlight for photosynthesis. However, in this example, the behavior of the system near the fixed point of the center type differs significantly from self-oscillations. In the example given, the projection of the motion of the phase point on the phase plane $O\Delta c_1 \Delta c_2$ although passing along closed curves, however, the amplitude of oscillations depends significantly on the initial conditions $\Delta c_{1(0)}$, $\Delta c_{2(0)}$ and is unstable to small perturbations. That is, after any small perturbation, the phase point begins to move along a new curve closed in the projection on the phase plane $O\Delta c_1 \Delta c_2$.

Construction of the phase portrait of the system of differential equations (2) can be quite a difficult problem since in the general case the differential equation [2]

$$\frac{dc_2}{dc_1} = \frac{f_2(c_1, c_2)}{f_1(c_1, c_2)}, \tag{24}$$

obtained from the system of differential equations (2) may not have an analytical solution. Then the construction of integral curves should be done by qualitative research. For this we can use the method of isoclines. Isocline lines on the phase plane intersect with all integral curves at the same angle with the *x*-axis, i.e. putting $\frac{dc_2}{dc_1} = k = \text{const}$ in the equation (24), we obtain

$$f_2(c_1, c_2) = kf_1(c_1, c_2) \quad \text{or} \quad f_1(c_1, c_2) = \frac{f_2(c_1, c_2)}{k} \tag{25}$$

is an equation of the isocline family.

We obtain the main isoclines (isoclines of horizontal and vertical tangent lines) for $k = \frac{dc_2}{dc_1} = 0$ and $k = \frac{dc_2}{dc_1} \to \infty$. Then, respectively, the equations of these isoclines according to the formulas (25) have the form:

$$f_2(c_1, c_2) = 0, \tag{26}$$

$$f_1(c_1, c_2) = 0, \tag{27}$$

where (26) is the isocline equation at the points of which the integral curves have horizontal tangent lines; (27) is the isocline equation at the points of which the integral curves have vertical tangent lines.

If we draw the straight line, through each point $(c_1; c_2) \in D$, that the tangent of the angle of inclination to the axis $O'c_1$ is equal to the right-hand part of the differential equation (24), then we obtain the family of straight lines that is the field of directions of the differential equation (24). The direction of the field at each point $(c_1; c_2) \in D$ is represented by a small segment of the straight line corresponding to this point. The center of each segment is selected at the point $(c_1; c_2)$. At each point of the isocline, the direction of the field is the same.

There are fixed points of the system of differential equations (2) at the intersection of the curves (26) and (27). We construct curves of isoclines (26) and (27) on the phase plane $O'c_1c_2$. In Fig. 5, the direction of the field at the points lying on the isoclines $f_1(c_1, c_2) = 0$ and $f_2(c_1, c_2) = 0$ is depicted by vertical and horizontal small segments, respectively (by linear elements of straight lines). At the points of the isocline (26), tangent lines to the integral curves are parallel to the axes of the abscissa, and at the points of the isocline (27), tangent lines to the integral curves are parallel to the axes of the ordinates. Using isoclines, we approximately construct curves that at each of their points touch the direction of the field at this point (in this case, there are only points of two isoclines $f_1(c_1, c_2) = 0$ and $f_2(c_1, c_2) = 0$). These curves are the integral curves $c_2 = c_2(c_1, C)$ of the differential equation (24) and the phase trajectories of the system of differential equations (2). Having constructed several isoclines (in this case, two isoclines (26) and (27)), and the field of directions on these isoclines, we approximately depict the integral curves of the differential equation (24), which coincide with the phase trajectories of the system of differential equations (2). For example, in Fig. 5, we see that under some conditions, depending on the type of functions $f_1(c_1, c_2)$, $f_2(c_1, c_2)$ and their arguments $c_1(t), c_2(t)$, the phase trajectories of the system of differential equations (2) go from the neighborhood of the fixed point $(\overline{c}_1; \overline{c}_2)$ (equilibrium position) in spirals, i.e. the phase curves are spirals that spin from the neighborhood of the fixed point.

Note that the direction of motion can be determined, for example, as follows. Consider in Fig.5 part of the phase plane $O'c_1c_2$, where indicated $k = \frac{dc_2}{dc_1} > 0$. Obviously, if in the specified part of the phase plane $O'c_1c_2$ $\frac{dc_2}{dt} > 0$ and $\frac{dc_1}{dt} > 0$ (then $k > 0$, since $\frac{\frac{dc_2}{dt}}{\frac{dc_1}{dt}} = \frac{dc_2}{dc_1} > 0$), then we put the arrow in the direction around the fixed point (Fig. 5). If the conditions $\frac{dc_2}{dt} < 0$ and $\frac{dc_1}{dt} < 0$ were satisfied, then the condition $k > 0$ would also be satisfied, but the arrow would be placed in the direction of the fixed point. In this case, the phase trajectories of the system of differential equations (2) would go to (into the neighborhood of) the fixed point $(\overline{c}_1;\overline{c}_2)$ (the equilibrium position) on the spirals, i.e. spirals, that are wound on the fixed point $(\overline{c}_1;\overline{c}_2)$, would be the phase curves.

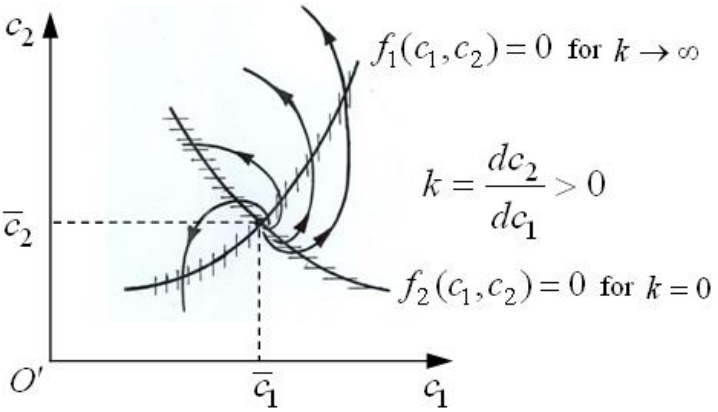

Fig. 5. **The phase trajectories of the system (2)**

Conclusions. Mathematical study of the stability of fixed points of systems of differential equations describing biochemical processes rates is performed in the article. The definition of the nature of the stability of fixed points is considered and investigated.

Bibliography

1. Hubal H.M. Mathematical Modeling of Biochemical Processes Rates in Biological Systems / H.M. Hubal // Computer-Integrated Technologies: Education, Science, Production. – 2021. – No. 42. – P. 43-49.

2. Hubal H.M. Mathematical Analysis of Qualitative Characteristics of Solutions of Systems of Differential Equations Describing Biochemical Processes Rates / H.M. Hubal // Міжвузівський збірник наукових праць "Наукові нотатки" за галузями знань "Фізико-математичні науки" та "Технічні науки" (за науковою спеціальністю 113 Прикладна математика). – 2021. – № 71. – С. 105–112.

3. Perko L. Differential Equations and Dynamical Systems / L. Perko. – 3rd ed. – Springer-Verlag, 2001. – 556 p.

4. Zill D.G. A First Course in Differential Equations with Modeling Applications / D. G. Zill. – 11th ed. – Cengage Learning, 2017. – 489 p.

5. Christopher C. Limit Cycles of Differential Equations / C. Christopher, Chengzhi Li, J. Torregrosa. – 2nd ed. – Birkhäuser Basel, 2021. – 171 p.

6. Cronin J. Ordinary Differential Equations: Introduction and Qualitative Theory / J. Cronin. – 3rd ed. – CRC Press, 2019. – 408 p.

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